GEOMETRY

Mr. Borger

Please read pages 420-433 in your textbook. Then do exercises 1-13 on page 422-23 and exercises 1-21 on page 432-3. Please write your answers on a separate sheet.

Your final grade depends on these exercises. A passing grade will be required to receive a regent diploma.

For those without the textbook, I have scanned and attached the appropriate pages. Arrangements will be made to pick up books. You will be notified iy"h in the near future.

You may return your answer sheet to:

mathb.mirrer@gmail.com

Or

FAX to 718 375 6342

Or

Mail to MIRRER HIGH SCHOOL

1791-5 Ocean Parkway

Brooklyn NY 11223

YOU MUST INDICATE ON YOUR ANSWER SHEET YOUR INFORMATION AS TO HOW YOU WOULD LIKE YOUR WORK TO BE RETURNED.

DEFINITION

Skew lines are lines in space that are neither parallel nor intersecting.

EXAMPLE 1

Does a triangle determine a plane?

Solution A triangle consists of three line segments and the three non-collinear points that are the endpoints of each pair of segments. Three non-collinear points determine a plane. That plane contains all of the points of the lines determined by the points. Therefore, a triangle determines a plane.

Exercises

Writing About Mathematics

1. Joel said that another definition for skew lines could be two lines that do not lie in the same plane. Do you agree with Joel? Explain why or why not.

2. Angelina said that if $\overline{AC}$ and $\overline{BD}$ intersect at a point, then A, B, C, and D lie in a plane and form a quadrilateral. Do you agree with Angelina? Explain why or why not.

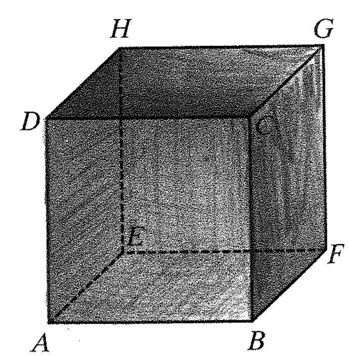
Developing Skills

3. $\overleftrightarrow{AB}$ is parallel to $\overleftrightarrow{CD}$ and $AB \neq CD$. Prove that A, B, C, and D must lie in a plane and form a trapezoid.

4. $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$ and $\overleftrightarrow{AD} \parallel \overleftrightarrow{BC}$. Prove that A, B, C, and D must lie in a plane and form a parallelogram.

5. $\overline{AEC} \cong \overline{BED}$ and each segment is the perpendicular bisector of the other. Prove that A, B, C, and D must lie in a plane and form a square.

In 6–9, use the diagram at the right.



6. Name two pairs of intersecting lines.

7. Name two pairs of skew lines.

8. Name two pairs of parallel lines.

9. Which pairs of lines that you named in exercises 6, 7, and 8 are not lines in the same plane?

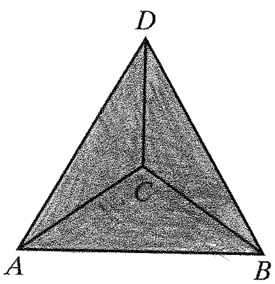
10. Let p represent "Two lines are parallel."
Let q represent "Two lines are coplanar."
Let r represent "Two lines have no point in common."
   a. Write the biconditional "Two lines are parallel if and only if they are coplanar and have no points in common" in terms of p, q, r, and logical symbols.
   b. The biconditional is true. Show that q is true when p is true.

Applying Skills

11. A photographer wants to have a steady base for his camera. Should he choose a base with four legs or with three? Explain your answer.

12. Ken is building a tool shed in his backyard. He begins by driving four stakes into the ground to be the corners of a rectangular floor. He stretches strings from two of the stakes to the opposite stakes and adjusts the height of the stakes until the strings intersect. Explain how the strings assure him that the four stakes will all be in the same plane.

13. Each of four equilateral triangles has a common side with each of the three other triangles and form a solid called a *tetrahedron*. Prove that the triangles are congruent.

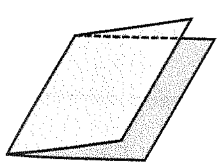


11-2 PERPENDICULAR LINES AND PLANES

Look at the floor, walls, and ceiling of the classroom. Each of these surfaces can be represented by a plane. Many of these planes intersect. For example, each wall intersects the ceiling and each wall intersects the floor. Each intersection can be represented by a line segment. This observation allows us to state the following postulate.

Postulate 11.3 If two planes intersect, then they intersect in exactly one line.

The Angle Formed by Two Intersecting Planes



Fold a piece of paper. The part of the paper on one side of the crease represents a half-plane and the crease represents the edge of the half-plane. The folded paper forms a dihedral angle.

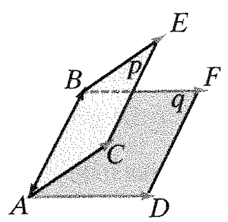
DEFINITION

A **dihedral angle** is the union of two half-planes with a common edge.

Each half-plane of a dihedral angle can be compared to a ray of an angle in a plane (or a **plane angle**) and the edge to the vertex of a plane angle. If we choose some point on the edge of a dihedral angle and draw, from this point, a ray in each half-plane perpendicular to the edge, we have drawn a plane angle.

DEFINITION

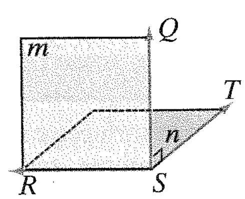
The **measure of a dihedral angle** is the measure of the plane angle formed by two rays each in a different half-plane of the angle and each perpendicular to the common edge at the same point of the edge.



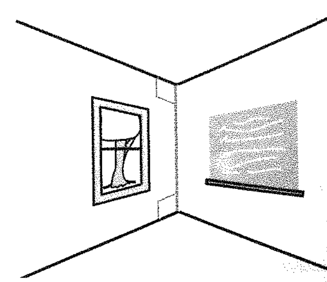
A plane angle whose measure is the same as that of the dihedral angle can be drawn at any points on the edge of the dihedral angle. Each plane angle of a dihedral angle has the same measure. In the figure, planes p and q intersect at $\overleftrightarrow{AB}$. In plane p, $\overrightarrow{AC} \perp \overleftrightarrow{AB}$ and in plane q, $\overrightarrow{AD} \perp \overleftrightarrow{AB}$. The measure of the dihedral angle is equal to the measure of $\angle CAD$. Also, in plane p, $\overrightarrow{BE} \perp \overleftrightarrow{AB}$ and in plane q, $\overrightarrow{BF} \perp \overleftrightarrow{AB}$. The measure of the dihedral angle is equal to the measure of $\angle EBF$.

DEFINITION

Perpendicular planes are two planes that intersect to form a right dihedral angle.



In the diagram, $\angle QST$ is a right angle. In plane m, $\overrightarrow{SQ} \perp \overrightarrow{SR}$ and in plane n, $\overrightarrow{ST} \perp \overrightarrow{SR}$. The dihedral angle formed by half-planes of planes m and n with edge $\overleftrightarrow{RS}$ has the same measure as $\angle QST$. Therefore, the dihedral angle is a right angle, and $m \perp n$.



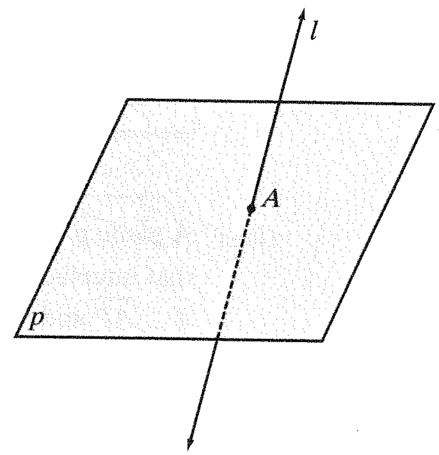
The floor and a wall of a room usually form a right dihedral angle. Look at the line that is the intersection of two adjacent walls of the classroom. This line intersects the ceiling in one point and intersects the floor in one point. This observation suggests the following theorem.

Theorem 11.3 If a line not in a plane intersects the plane, then it intersects in exactly one point.

Given Line l is not in plane p and l intersects p.

Prove Line l intersects p in exactly one point.

Proof Use an indirect proof.



Assume that line l intersects the plane in two points. Then all of the points on line l lie in plane p, that is, the line lies in the plane. Because this contradicts the hypothesis that line l is not in plane p, the assumption must be false. A line not in a plane that intersects the plane, intersects it in exactly one point.

Again, look at the corner of the classroom in the figure on the previous page. The line that is the intersection of two adjacent walls intersects the ceiling so that the line is perpendicular to any line in the ceiling through the point of intersection. We say that this line is perpendicular to the plane of the ceiling.

DEFINITION

A line is perpendicular to a plane if and only if it is perpendicular to each line in the plane through the intersection of the line and the plane.

A **plane is perpendicular to a line** if the line is perpendicular to the plane.

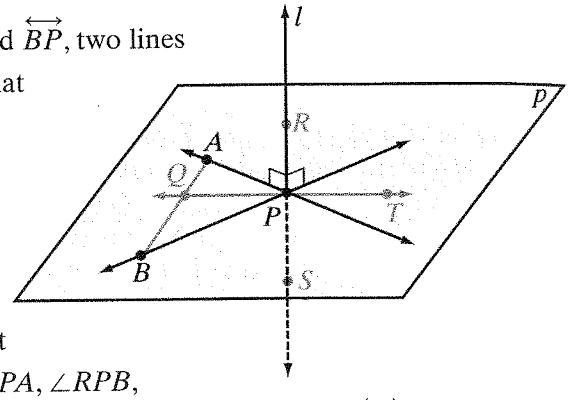
It is easy to demonstrate that a line that is perpendicular to one line in a plane may not be perpendicular to the plane. For example, fold a rectangular sheet of paper. Draw a ray perpendicular to the crease with its endpoint on the crease. Keep the half of the folded sheet that does not contain the ray in contact with your desk. This is the plane. Move the other half to different positions. The ray that you drew is always perpendicular to the crease but is not always perpendicular to the plane. Based on this observation, we can state the following postulate.

Postulate 11.4 At a given point on a line, there are infinitely many lines perpendicular to the given line.

In order to prove that a line is perpendicular to a plane, the definition requires that we show that *every* line through the point of intersection is perpendicular to the given line. However, it is possible to prove that if line l is known to be perpendicular to each of two lines in plane p that intersect at point A, then l is perpendicular to plane p at A.

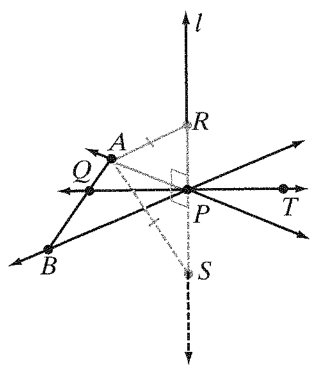
Theorem 11.4 If a line is perpendicular to each of two intersecting lines at their point of intersection, then the line is perpendicular to the plane determined by these lines.

Given A plane p determined by $\overleftrightarrow{AP}$ and $\overleftrightarrow{BP}$, two lines that intersect at P. Line l such that $l \perp \overleftrightarrow{AP}$ and $l \perp \overleftrightarrow{BP}$.

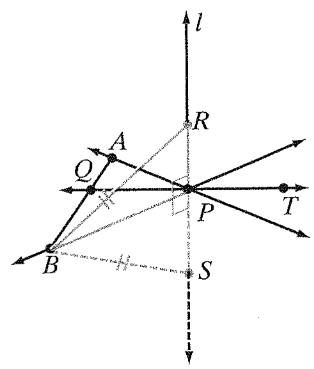


Prove $l \perp p$

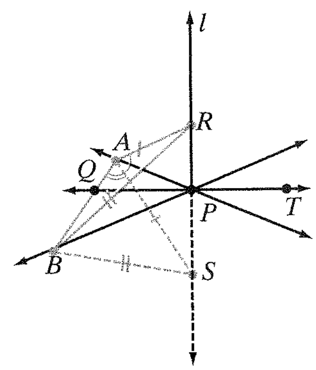
Proof To begin, let R and S be points on l such that P is the midpoint of $\overline{RS}$. Since it is given that $l \perp \overleftrightarrow{AP}$ and $l \perp \overleftrightarrow{BP}$, $\angle RPA$, $\angle SPA$, $\angle RPB$, and $\angle SPB$ are right angles and therefore congruent. Then let $\overleftrightarrow{PT}$ be any other line through P in plane p. Draw $\overleftrightarrow{AB}$ intersecting $\overleftrightarrow{PT}$ at Q. To prove this theorem, we need to show three different pairs of congruent triangles: $\triangle RPA \cong \triangle SPA$, $\triangle RPB \cong \triangle SPB$, and $\triangle RPQ \cong \triangle SPQ$. However, to establish the last congruence we must prove that $\triangle RAB \cong \triangle SAB$ and $\triangle RAQ \cong \triangle SAQ$.



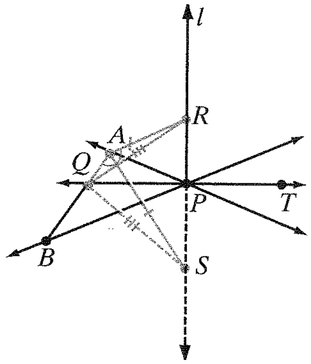
(1) $\triangle RPA \cong \triangle SPA$ by SAS and $\overline{AR} \cong \overline{AS}$.



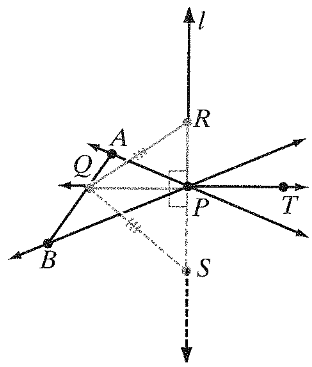
(2) $\triangle RPB \cong \triangle SPB$ by SAS and $\overline{BR} \cong \overline{BS}$.



(3) $\triangle RAB \cong \triangle SAB$ by SSS and $\angle RAB \cong \angle SAB$.



(4) $\triangle RAQ \cong \triangle SAQ$ by by SAS since $\angle RAB \cong \angle RAQ$ and $\angle SAB \cong \angle SAQ$, and $\overline{RQ} \cong \overline{SQ}$.



(5) $\triangle RPQ \cong \triangle SPQ$ by SSS and $\angle RPQ \cong \angle SPQ$.

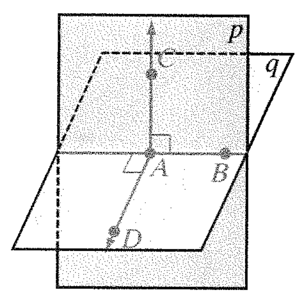
Now since $\angle RPQ$ and $\angle SPQ$ are a congruent linear pair of angles, they are right angles, and $l \perp \overleftrightarrow{PQ}$. Since $\overleftrightarrow{PQ}$, that is, $\overleftrightarrow{PT}$ can be *any* line in p through P, l is perpendicular to *every* line in plane p through point P. ∎

Theorem 11.5a

> If two planes are perpendicular to each other, one plane contains a line perpendicular to the other plane.

Given Plane $p \perp$ plane q

Prove A line in p is perpendicular to q and a line in q is perpendicular to p.



Proof If planes p and q are perpendicular to each other then they form a right dihedral angle. Let $\overleftrightarrow{AB}$ be the edge of the dihedral angle. In plane p, construct $\overleftrightarrow{AC} \perp \overleftrightarrow{AB}$, and in plane q, construct $\overleftrightarrow{AD} \perp \overleftrightarrow{AB}$. Since p and q from a right dihedral angle, $\angle CAD$ is a right angle and $\overleftrightarrow{AC} \perp \overleftrightarrow{AD}$. Two lines in plane p, $\overleftrightarrow{AB}$ and $\overleftrightarrow{AC}$, are each perpendicular to $\overleftrightarrow{AD}$. Therefore, $\overleftrightarrow{AD} \perp p$. Similarly, two lines in plane q, $\overleftrightarrow{AB}$ and $\overleftrightarrow{AD}$, are each perpendicular to $\overleftrightarrow{AC}$. Therefore, $\overleftrightarrow{AC} \perp q$. ∎

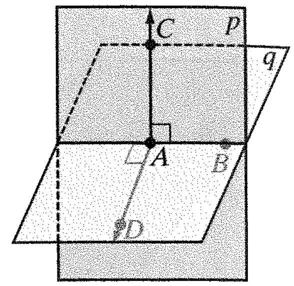
The converse of Theorem 11.5a is also true.

Theorem 11.5b

If a plane contains a line perpendicular to another plane, then the planes are perpendicular.

Given $\overleftrightarrow{AC}$ in plane p and $\overleftrightarrow{AC} \perp q$

Prove $p \perp q$



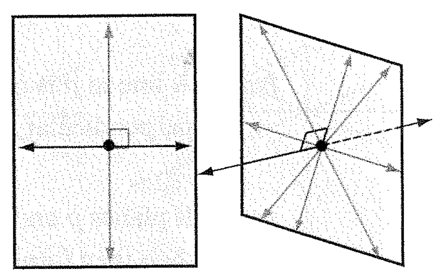
Proof Let $\overleftrightarrow{AB}$ be the line of intersection of planes p and q. In plane q, draw $\overleftrightarrow{AD} \perp \overleftrightarrow{AB}$. Since $\overleftrightarrow{AC}$ is perpendicular to q, it is perpendicular to any line through A in q. Therefore, $\overleftrightarrow{AC} \perp \overleftrightarrow{AD}$ and also $\overleftrightarrow{AC} \perp \overleftrightarrow{AB}$. Thus, $\angle CAD$ is the plane angle of the dihedral angle formed by planes p and q. Since $\overleftrightarrow{AC}$ is perpendicular to $\overleftrightarrow{AD}$, $\angle CAD$ is a right angle. Therefore, the dihedral angle is a right angle, and p and q are perpendicular planes.

This theorem and its converse can be stated as a biconditional.

Theorem 11.5

Two planes are perpendicular if and only if one plane contains a line perpendicular to the other.

We know that in a plane, there is only one line perpendicular to a given line at a given point on the line. In space, there are infinitely many lines perpendicular to a given line at a given point on the line. These perpendicular lines are all in the same plane. However, only one line is perpendicular to a plane at a given point.

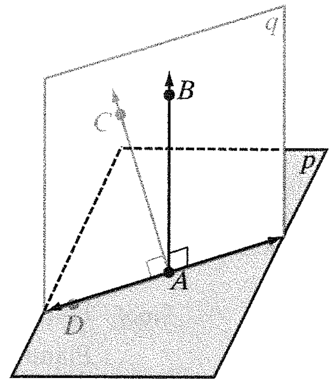


Theorem 11.6

Through a given point on a plane, there is only one line perpendicular to the given plane.

Given Plane p and $\overleftrightarrow{AB} \perp p$ at A.

Prove $\overleftrightarrow{AB}$ is the only line perpendicular to p at A.



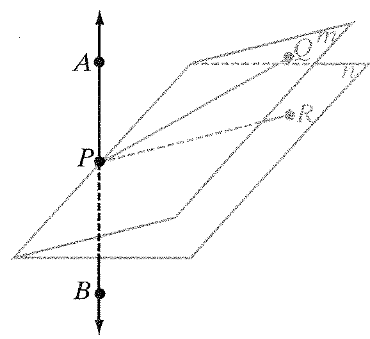
Proof Use an indirect proof.
Assume that there exists another line, $\overleftrightarrow{AC} \perp p$ at A. Points A, B, and C determine a plane, q, that intersects plane p at $\overleftrightarrow{AD}$. Therefore, in plane q, $\overleftrightarrow{AB} \perp \overleftrightarrow{AD}$ and $\overleftrightarrow{AC} \perp \overleftrightarrow{AD}$. But in a given plane, there is only one line perpendicular to a given line at a given point. Our assumption is false, and there is only one line perpendicular to a given plane at a given point.

As we noted above, in space, there are infinitely many lines perpendicular to a given line at a given point. Any two of those intersecting lines determine a plane perpendicular to the given line. Each of these pairs of lines determine the same plane perpendicular to the given line.

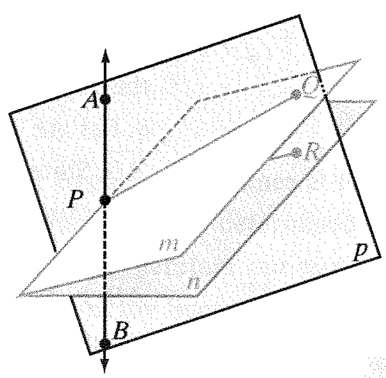
Theorem 11.7 Through a given point on a line, there can be only one plane perpendicular to the given line.

Given Any point P on $\overleftrightarrow{AB}$.

Prove There is only one plane perpendicular to $\overleftrightarrow{AB}$.



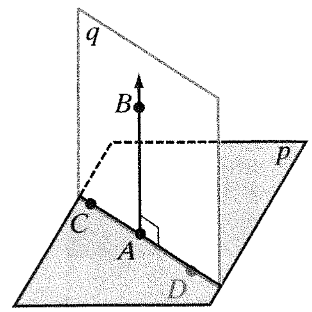
Proof Use an indirect proof.
Assume that there are two planes, m and n, that are each perpendicular to $\overleftrightarrow{AB}$. Choose any point Q in m. Since $m \perp \overleftrightarrow{APB}$, $\overleftrightarrow{AP} \perp \overleftrightarrow{PQ}$. Points A, P, and Q determine a plane p that intersects plane n in a line $\overleftrightarrow{PR}$. Since $n \perp \overleftrightarrow{APB}$, $\overleftrightarrow{AP} \perp \overleftrightarrow{PR}$. Therefore, in plane p, $\overleftrightarrow{AP} \perp \overleftrightarrow{PQ}$ and $\overleftrightarrow{AP} \perp \overleftrightarrow{PR}$. But in a plane, at a given point there is one and only one line perpendicular to a given line. Our assumption must be false, and there is only one plane perpendicular to $\overleftrightarrow{AB}$ at P.



Theorem 11.8 If a line is perpendicular to a plane, then any line perpendicular to the given line at its point of intersection with the given plane is in the plane.

Given $\overleftrightarrow{AB} \perp$ plane p at A and $\overleftrightarrow{AB} \perp \overleftrightarrow{AC}$.

Prove $\overleftrightarrow{AC}$ is in plane p.

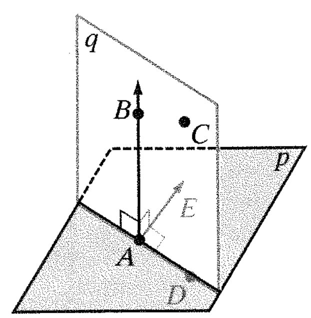


Proof Points A, B, and C determine a plane q. Plane q intersects plane p in a line, $\overleftrightarrow{AD}$. $\overleftrightarrow{AB} \perp \overleftrightarrow{AD}$ because $\overleftrightarrow{AB}$ is perpendicular to every line in p through A. It is given that $\overleftrightarrow{AB} \perp \overleftrightarrow{AC}$. Therefore, $\overleftrightarrow{AD}$ and $\overleftrightarrow{AC}$ in plane q are perpendicular to $\overleftrightarrow{AB}$ at A. But at a given point in a plane, only one line can be drawn perpendicular to a given line. Therefore, $\overleftrightarrow{AD}$ and $\overleftrightarrow{AC}$ are the same line, that is, C is on $\overleftrightarrow{AD}$. Since $\overleftrightarrow{AD}$ is the intersection of planes p and q, $\overleftrightarrow{AC}$ is in plane p.

Theorem 11.9 If a line is perpendicular to a plane, then every plane containing the line is perpendicular to the given plane.

Given Plane p with $\overleftrightarrow{AB} \perp p$ at A, and C any point not on p.

Prove The plane q determined by A, B, and C is perpendicular to p.



Proof Let the intersection of p and q be $\overleftrightarrow{AD}$, so $\overleftrightarrow{AD}$ is the edge of the dihedral angle formed by p and q. Let $\overleftrightarrow{AE}$ be a line in p that is perpendicular to $\overleftrightarrow{AD}$. Since $\overleftrightarrow{AB} \perp p$, $\overleftrightarrow{AB}$ is perpendicular to every line in p through A. Therefore, $\overleftrightarrow{AB} \perp \overleftrightarrow{AD}$ and $\overleftrightarrow{AB} \perp \overleftrightarrow{AE}$. $\angle BAE$ is a plane angle whose measure is the measure of the dihedral angle. Since $\overleftrightarrow{AB} \perp \overleftrightarrow{AE}$, $m\angle BAE = 90$. Therefore, the dihedral angle is a right angle, and $q \perp p$.

EXAMPLE 1

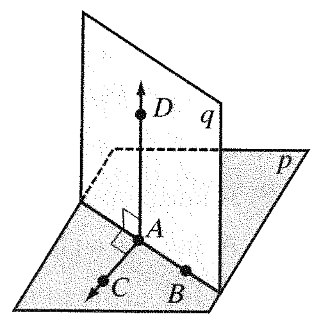
Show that the following statement is false.

Two planes perpendicular to the same plane have no points in common.

Solution Recall that a statement that is sometimes true and sometimes false is regarded to be false. Consider the adjacent walls of a room. Each wall is perpendicular to the floor but the walls intersect in a line. This counterexample shows that the given statement is false.

EXAMPLE 2

Planes p and q intersect in $\overleftrightarrow{AB}$. In p, $\overleftrightarrow{AC} \perp \overleftrightarrow{AB}$ and in q, $\overleftrightarrow{AD} \perp \overleftrightarrow{AB}$. If $m\angle CAD < 90$, is $p \perp q$?

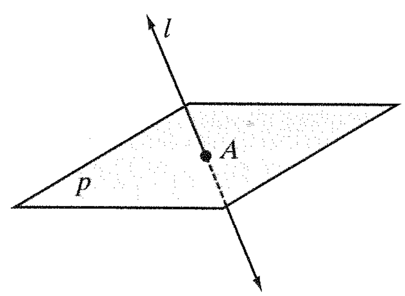


Solution Since in p, $\overleftrightarrow{AC} \perp \overleftrightarrow{AB}$, and in q, $\overleftrightarrow{AD} \perp \overleftrightarrow{AB}$, $\angle CAD$ is a plane angle whose measure is equal to the measure of the dihedral angle formed by the planes. Since $\angle CAD$ is not a right angle, then the dihedral angle is not a right angle, and the planes are not perpendicular.

EXAMPLE 3

Given: Line l intersects plane p at A, and l is not perpendicular to p.

Prove: There is at least one line through A in plane p that is not perpendicular to l.



Proof Use an indirect proof.

Let $\overleftrightarrow{AB}$ and $\overleftrightarrow{AC}$ be two lines through A in p. Assume that $l \perp \overleftrightarrow{AB}$ and $l \perp \overleftrightarrow{AC}$. Therefore, $l \perp p$ because if a line is perpendicular to each of two lines at their point of intersection, then the line is perpendicular to the plane determined by these lines. But it is given that l is not perpendicular to p. Therefore, our assumption is false, and l is not perpendicular to at least one of the lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{AC}$.

Exercises

Writing About Mathematics

1. Carmen said if two planes intersect to form four dihedral angles that have equal measures, then the planes are perpendicular to each other. Do you agree with Carmen? Explain why or why not.

2. Each of three lines is perpendicular to the plane determined by the other two.

   a. Is each line perpendicular to each of the other two lines? Justify your answer.

   b. Name a physical object that justifies your answer.

Developing Skills

In 3–11, state whether each of the statements is true or false. If it is true, state a postulate or theorem that supports your answer. If it is false, describe or draw a counterexample.

3. At a given point on a given line, only one line is perpendicular to the line.

4. If A is a point in plane p and B is a point not in p, then no other point on $\overleftrightarrow{AB}$ is in plane p.

5. A line perpendicular to a plane is perpendicular to every line in the plane.

6. A line and a plane perpendicular to the same line at two different points have no points in common.

7. Two intersecting planes that are each perpendicular to a third plane are perpendicular to each other.

8. If $\overleftrightarrow{AB}$ is perpendicular to plane p at A and $\overleftrightarrow{AB}$ is in plane q, then $p \perp q$.

9. At a given point on a given plane, only one plane is perpendicular to the given plane.

10. If a plane is perpendicular to one of two intersecting lines, it is perpendicular to the other.

11. If a line is perpendicular to one of two intersecting planes, it is perpendicular to the other.

Applying Skills

12. Prove step 1 of Theorem 11.4.

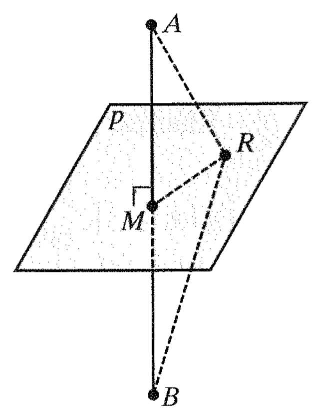
13. Prove step 3 of Theorem 11.4.

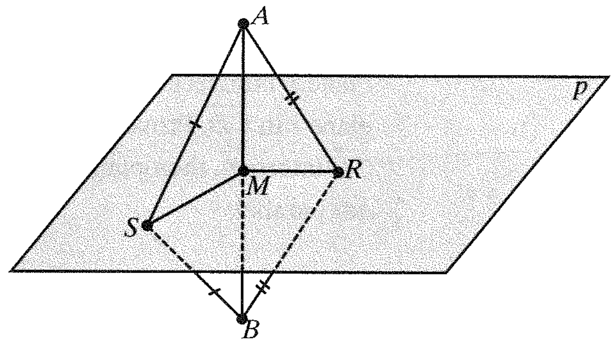
14. Prove step 5 of Theorem 11.4.

15. Prove that if a line segment is perpendicular to a plane at the midpoint of the line segment, then every point in the plane is equidistant from the endpoints of the line segment.

    Given: $\overline{AB} \perp$ plane p at M, the midpoint of $\overline{AB}$, and R is any point in plane p.

    Prove: $AR = BR$





16. Prove that if two points are each equidistant from the endpoints of a line segment, then the line segment is perpendicular to the plane determined by the two points and the midpoint of the line segment.

Given: M is the midpoint of $\overline{AB}$, $\overline{RA} \cong \overline{RB}$, and $\overline{SA} \cong \overline{SB}$.

Prove: $\overline{AB}$ is perpendicular to the plane determined by M, R, and S.

17. Equilateral triangle ABC is in plane p and $\overleftrightarrow{AD}$ is perpendicular to plane p. Prove that $\overline{BD} \cong \overline{CD}$.

18. $\overleftrightarrow{AB}$ and $\overleftrightarrow{AC}$ intersect at A and determine plane p. $\overleftrightarrow{AD}$ is perpendicular to plane p at A. If $AB = AC$, prove that $\triangle ABD \cong \triangle ACD$.

19. Triangle QRS is in plane p, $\overline{ST}$ is perpendicular to plane p, and $\angle QTS \cong \angle RTS$. Prove that $\overline{TQ} \cong \overline{TR}$.

20. Workers who are installing a new telephone pole position the pole so that it is perpendicular to the ground along two different lines. Prove that this is sufficient to prove that the telephone pole is perpendicular to the ground.

21. A telephone pole is perpendicular to the level ground. Prove that two wires of equal length attached to the pole at the same point and fastened to the ground are at equal distances from the pole.

11-3 PARALLEL LINES AND PLANES

Look at the floor, walls, and ceiling of the classroom. Each of these surfaces can be represented by a plane. Some of these surfaces, such as the floor and the ceiling, do not intersect. These can be represented as portions of *parallel planes*.

DEFINITION

Parallel planes are planes that have no points in common.

A **line is parallel to a plane** if it has no points in common with the plane.