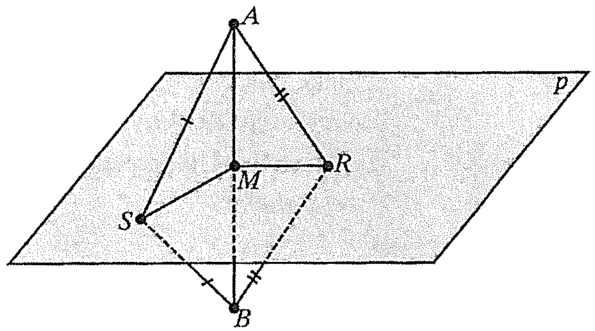
16. Prove that if two points are each equidistant from the endpoints of a line segment, then the line segment is perpendicular to the plane determined by the two points and the midpoint of the line segment.



Given: M is the midpoint of $\overline{AB}$, $\overline{RA} \cong \overline{RB}$, and $\overline{SA} \cong \overline{SB}$.

Prove: $\overline{AB}$ is perpendicular to the plane determined by M, R, and S.

17. Equilateral triangle ABC is in plane p and $\overleftrightarrow{AD}$ is perpendicular to plane p. Prove that $\overline{BD} \cong \overline{CD}$.

18. $\overleftrightarrow{AB}$ and $\overleftrightarrow{AC}$ intersect at A and determine plane p. $\overleftrightarrow{AD}$ is perpendicular to plane p at A. If $AB = AC$, prove that $\triangle ABD \cong \triangle ACD$.

19. Triangle QRS is in plane p, $\overline{ST}$ is perpendicular to plane p, and $\angle QTS \cong \angle RTS$. Prove that $\overline{TQ} \cong \overline{TR}$.

20. Workers who are installing a new telephone pole position the pole so that it is perpendicular to the ground along two different lines. Prove that this is sufficient to prove that the telephone pole is perpendicular to the ground.

21. A telephone pole is perpendicular to the level ground. Prove that two wires of equal length attached to the pole at the same point and fastened to the ground are at equal distances from the pole.

11-3 PARALLEL LINES AND PLANES

Look at the floor, walls, and ceiling of the classroom. Each of these surfaces can be represented by a plane. Some of these surfaces, such as the floor and the ceiling, do not intersect. These can be represented as portions of *parallel planes*.

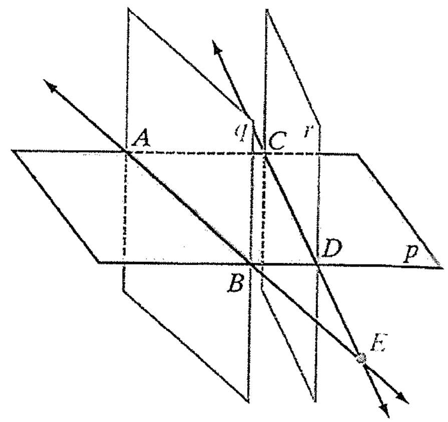
DEFINITION

Parallel planes are planes that have no points in common.

A **line is parallel to a plane** if it has no points in common with the plane.

EXAMPLE 1

Plane p intersects plane q in $\overleftrightarrow{AB}$ and plane r in $\overleftrightarrow{CD}$. Prove that if $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ intersect, then planes q and r are not parallel.



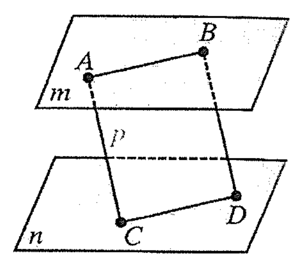
Proof Let E be the point at which $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ intersect. Then E is a point on q and E is a point on r. Therefore, planes q and r intersect in at least one point and are not parallel.

Theorem 11.10

If a plane intersects two parallel planes, then the intersection is two parallel lines.

Given Plane p intersects plane m at $\overleftrightarrow{AB}$ and plane n at $\overleftrightarrow{CD}$, $m \parallel n$.

Prove $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$



Proof Use an indirect proof.

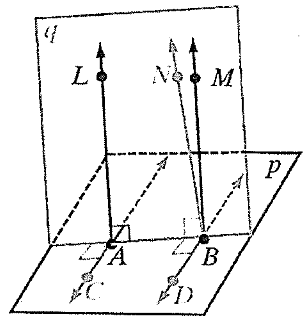
Lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are two lines of plane p. Two lines in the same plane either intersect or are parallel. If $\overleftrightarrow{AB}$ is not parallel to $\overleftrightarrow{CD}$ then they intersect in some point E. Since E is a point of $\overleftrightarrow{AB}$, then it is a point of plane m. Since E is a point of $\overleftrightarrow{CD}$, then it is a point of plane n. But $m \parallel n$ and have no points in common. Therefore, $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are two lines in the same plane that do not intersect, and $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$.

In a plane, two lines perpendicular to a given line are parallel. Can we prove that two lines perpendicular to a given plane are also parallel?

Theorem 11.11 Two lines perpendicular to the same plane are parallel.

Given Plane p, line $\overleftrightarrow{LA} \perp p$ at A, and line $\overleftrightarrow{MB} \perp p$ at B.

Prove $\overleftrightarrow{LA} \parallel \overleftrightarrow{MB}$



Proof We will construct line $\overleftrightarrow{NB}$ at B that is parallel to $\overleftrightarrow{LA}$, and show that $\overleftrightarrow{MB}$ and $\overleftrightarrow{NB}$ are the same line.

(1) Since it is given that $\overleftrightarrow{LA} \perp p$ at A, $\overleftrightarrow{LA}$ is perpendicular to any line in p through A, so $\overleftrightarrow{LA} \perp \overleftrightarrow{AB}$. Let q be the plane determined by $\overleftrightarrow{LA}$ and $\overleftrightarrow{AB}$. In plane p, draw $\overleftrightarrow{AC} \perp \overleftrightarrow{AB}$. Then $\angle LAC$ is a right angle, and p and q form a right dihedral angle.

(2) At point B, there exists a line $\overleftrightarrow{NB}$ in q that is parallel to $\overleftrightarrow{LA}$. If one of two parallel lines is perpendicular to a third line, then the other is perpendicular to the third line, that is, since $\overleftrightarrow{LA} \perp \overleftrightarrow{AB}$, then $\overleftrightarrow{NB} \perp \overleftrightarrow{AB}$.

(3) Draw $\overleftrightarrow{BD} \perp \overleftrightarrow{AB}$ in p. Because p and q form a right dihedral angle, $\angle NBD$ is a right angle, and so $\overleftrightarrow{NB} \perp \overleftrightarrow{BD}$.

(4) Therefore, $\overleftrightarrow{NB}$ is perpendicular to two lines in p at B (steps 2 and 3), so $\overleftrightarrow{NB}$ is perpendicular to p at B.

(5) But it is given that $\overleftrightarrow{MB} \perp p$ at B and there is only one line perpendicular to a given plane at a given point. Therefore, $\overleftrightarrow{MB}$ and $\overleftrightarrow{NB}$ are the same line, and $\overleftrightarrow{LA} \parallel \overleftrightarrow{MB}$. ┘

We have shown that two lines perpendicular to the same plane are parallel. Since parallel lines lie in the same plane, we have just proved the following corollary to this theorem:

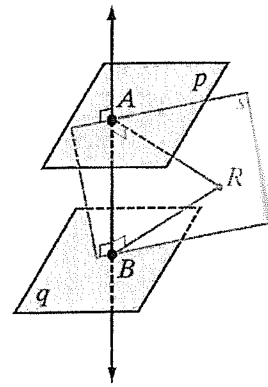
Corollary 11.11a Two lines perpendicular to the same plane are coplanar.

Theorem 11.12a If two planes are perpendicular to the same line, then they are parallel.

Given Plane $p \perp \overleftrightarrow{AB}$ at A and $q \perp \overleftrightarrow{AB}$ at B.

Prove $p \parallel q$

Proof Use an indirect proof.

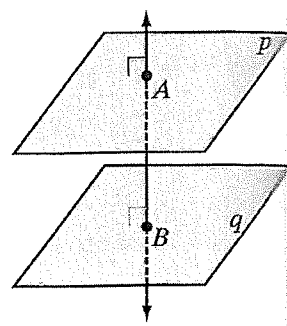


Assume that p is not parallel to q. Then p and q intersect in a line. Let R be any point on the line of intersection. Then A, B, and R determine a plane, s. In plane s, $\overleftrightarrow{AR} \perp \overleftrightarrow{AB}$ and $\overleftrightarrow{BR} \perp \overleftrightarrow{AB}$. But two lines in a plane that are perpendicular to the same line are parallel. Therefore, our assumption must be false, and $p \parallel q$.

Theorem 11.12b If two planes are parallel, then a line perpendicular to one of the planes is perpendicular to the other.

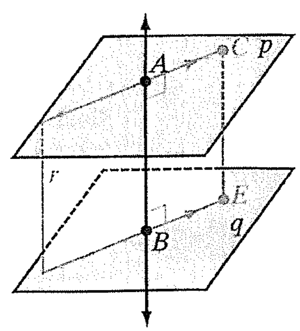
Given Plane p parallel to plane q, and $\overleftrightarrow{AB} \perp$ plane p and intersecting plane q at B

Prove $\overleftrightarrow{AB} \perp$ plane q

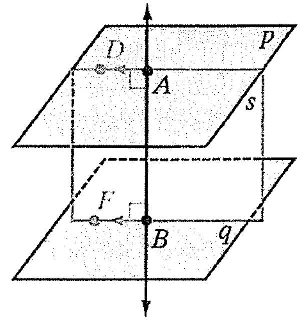


Proof To prove this theorem, we will construct two lines $\overleftrightarrow{BE}$ and $\overleftrightarrow{FB}$ in q that are both perpendicular to $\overleftrightarrow{AB}$. From this, we will conclude that $\overleftrightarrow{AB}$ is perpendicular to q.

(1) Let C be a point in p. Let r be the plane determined by A, B, and C intersecting q at $\overleftrightarrow{BE}$. Since p and q are parallel, $\overleftrightarrow{AC} \parallel \overleftrightarrow{BE}$. It is given that $\overleftrightarrow{AB} \perp$ plane p. Therefore, $\overleftrightarrow{AB} \perp \overleftrightarrow{AC}$. Then, in plane r, $\overleftrightarrow{AB} \perp \overleftrightarrow{BE}$.



(2) Let D be a point in p. Let s be the plane determined by A, B, and D intersecting q at $\overleftrightarrow{BF}$. Since p and q are parallel, $\overleftrightarrow{AD} \parallel \overleftrightarrow{BF}$. It is given that $\overleftrightarrow{AB} \perp$ plane p. Therefore, $\overleftrightarrow{AB} \perp \overleftrightarrow{AD}$. Then, in plane s, $\overleftrightarrow{AB} \perp \overleftrightarrow{BF}$.



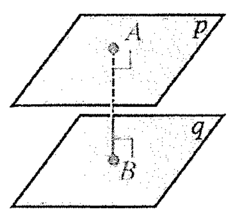
(3) If a line is perpendicular to each of two intersecting lines at their point of intersection, then the line is perpendicular to the plane determined by these lines. Therefore, $\overleftrightarrow{AB} \perp$ plane q.

Theorem 11.12a and 11.2b are converse statements. Therefore, we may write these two theorems as a biconditional.

Theorem 11.12

> Two planes are perpendicular to the same line if and only if the planes are parallel.

Let p and q be two parallel planes. From A in p, draw $\overline{AB} \perp q$ at B. Therefore, $\overline{AB} \perp p$ at A. The distance from p to q is AB.



DEFINITION

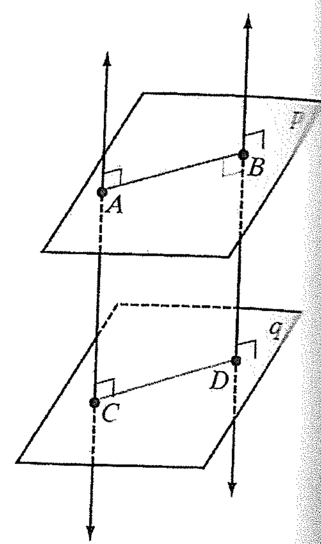
The **distance between two planes** is the length of the line segment perpendicular to both planes with an endpoint on each plane.

Theorem 11.13 Parallel planes are everywhere equidistant.

Given Parallel planes p and q, with $\overline{AC}$ and $\overline{BD}$ each perpendicular to p and q with an endpoint on each plane.

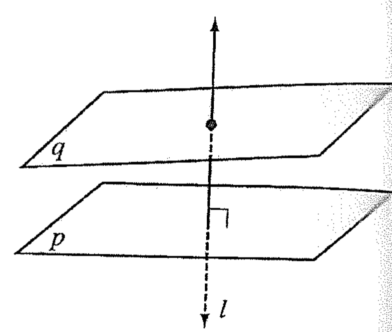
Prove $AC = BD$

Proof Two lines perpendicular to the same plane are both parallel and coplanar. Therefore, $\overline{AC} \parallel \overline{BD}$ and lie on the same plane. That plane intersects parallel planes p and q in parallel lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$. In the plane of $\overleftrightarrow{AC}$ and $\overleftrightarrow{BD}$, $ABDC$ is a parallelogram with a right angle, that is, a rectangle. Therefore, $\overline{AC}$ and $\overline{BD}$ are congruent and $AC = BD$.



EXAMPLE 2

Line l is perpendicular to plane p and line l is not perpendicular to plane q. Is $p \parallel q$?

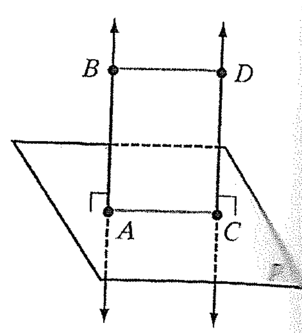


Solution Assume that $p \parallel q$. If two planes are parallel, then a line perpendicular to one is perpendicular to the other. Therefore, since l is perpendicular to plane p, l must be perpendicular to plane q. This contradicts the given statement that l is not perpendicular to q, and the assumption is false. Therefore, p is not parallel to q.

EXAMPLE 3

Given: $\overleftrightarrow{AB} \perp$ plane p at A, $\overleftrightarrow{CD} \perp$ plane p at C, and $AB = CD$.

Prove: A, B, C, and D are the vertices of a parallelogram.



Proof Two lines perpendicular to the same plane are parallel and coplanar. Therefore, since it is given that $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are each perpendicular to p, they are parallel and coplanar. Since $AB = CD$ and segments of equal length are congruent, $ABCD$ is a quadrilateral with one pair of sides congruent and parallel. Therefore, $ABCD$ is a parallelogram. ┘

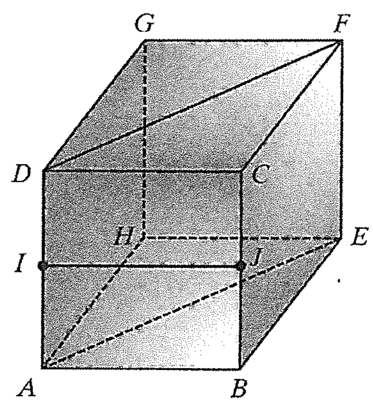
Exercises

Writing About Mathematics

1. Two planes are perpendicular to the same plane. Are the planes parallel? Justify your answer.
2. Two planes are parallel to the same plane. Are the planes parallel? Justify your answer.

Developing Skills

In 3–9, each of the given statements is sometimes true and sometimes false. **a.** Give an example from the diagram to show that the statement can be true. **b.** Give a counterexample from the diagram to show that the statement can be false. In the diagram, each quadrilateral is a rectangle.



3. If two planes are perpendicular, a line parallel to one plane is perpendicular to the other.
4. Two planes parallel to the same line are parallel to each other.
5. Two lines perpendicular to the same line are parallel.
6. Two lines that do not intersect are parallel.
7. Two planes perpendicular to the same plane are parallel to each other.
8. Two lines parallel to the same plane are parallel.
9. If two lines are parallel, then a line that is skew to one line is skew to the other.

Applying Skills

10. ABC is an isosceles triangle with base $\overline{BC}$ in plane p. Plane $q \parallel p$ through point D on $\overline{AB}$ and point E on $\overline{AC}$. Prove that $\triangle ADE$ is isosceles.
11. Plane p is perpendicular to $\overleftrightarrow{PQ}$ at Q and two points in p, A and B, are equidistant from P. Prove that $\overline{AQ} \cong \overline{BQ}$.

12. Noah is building a tool shed. He has a rectangular floor in place and wants to be sure that the posts that he erects at each corner of the floor as the ends of the walls are parallel. He erects each post perpendicular to the floor. Are the posts parallel to each other? Justify your answer.

13. Noah wants the flat ceiling on his tool shed to be parallel to the floor. Two of the posts are 80 inches long and two are 78 inches long. Will the ceiling be parallel to the floor? Justify your answer. What must Noah do to make the ceiling parallel to the floor?

11-4 SURFACE AREA OF A PRISM

Polyhedron

In the plane, a polygon is a closed figure that is the union of line segments. In space, a *polyhedron* is a figure that is the union of polygons.

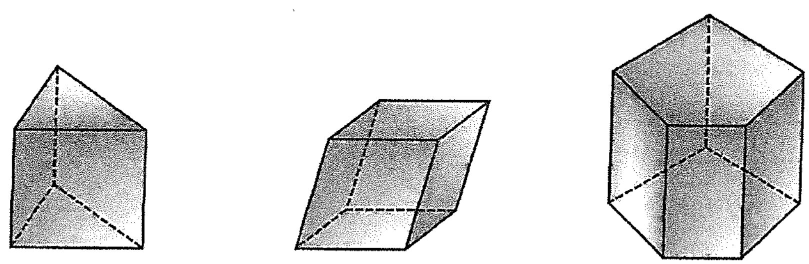
DEFINITION

A **polyhedron** is a three-dimensional figure formed by the union of the surfaces enclosed by plane figures.

The portions of the planes enclosed by a plane figure are called the **faces** of the polyhedron. The intersections of the faces are the **edges** of the polyhedron and the intersections of the edges are the **vertices** of the polyhedron.

DEFINITION

A **prism** is a polyhedron in which two of the faces, called the **bases** of the prism, are congruent polygons in parallel planes.



Examples of prisms

The surfaces between corresponding sides of the bases are called the **lateral sides** of the prism and the common edges of the lateral sides are called the **lateral edges**. An **altitude** of a prism is a line segment perpendicular to each of the bases with an endpoint on each base. The **height** of a prism is the length of an altitude.